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Boundary Terms and Noether Current of Spherical Black Holes

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Abstract

We consider two proposals for defining black hole entropy in spherical symmetry, where the horizon is defined locally as a trapping horizon. The first case, boundary terms in a dual-null form of the reduced action in two dimensions, gives a result that is proportional to the area. The second case, Wald's Noether current method, is generalized to dynamic black holes, giving an entropy that is just the area of the trapping horizon. These results are compared with a generalized first law of thermodynamics.

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1 Introduction

There has been much interest in the problem of black hole entropy, ever since the famous argument of Bekenstein was introduced to save the second law of thermodynamics: in order for entropy not be lost into a black hole, the black hole must have entropy. Then, Hawking [1] discovered that black holes radiate with a black body spectrum at a temperature of $T = \kappa/2\pi$ where κ is the surface gravity at the horizon. These two discoveries led to the study of black hole thermodynamics. In particular, stationary black holes were found [2] to have a first law or energy balance equation $\delta m = \kappa\delta A/8\pi$ plus work terms. To date, most of the work in this field has been done for stationary or (in some sense) quasi-stationary black holes. This should properly be thought of as thermostatics and not thermodynamics. The dynamical case contains many technological and definitional problems which are recently starting to be resolved. For example, a generalized first law of thermodynamics has been suggested in [3].

However, black hole thermodynamics and thermostatics have a footing only in the classical regime. There is as yet no clear statistical mechanical origin for entropy of a black hole. Do the states that make up this entropy lie inside the black hole, throughout the space-time, on the event horizon, or on a local horizon? What quantum states compose this entropy? Do these states preserve the "information" collected in the black holes by given correlation functions with the radiated energy? Is there a remnant quanta at the end of evaporation? As such, black hole entropy is one of the leading testing grounds for quantum gravity. One of the key tests to a successful quantum theory is the ability to predict the area law, which to date has been tested in various cases in the classical limit. There have been many suggestions as to where this entropy might come from and ways of calculating it: Noether current [4], D-branes [5], spin systems [6], entanglements [7], and surface terms that break the symmetry of the system [8] [9]. Each of these areas shows promising results but many of them contain limitations. For example, the Noether current needs a bifurcation surface in the system to define the entropy [4]. In the case of boundary terms, the only quantum calculations done to date have been in 2+1 dimensions [10] and also rely heavily on string techniques that are difficult generalize to higher dimensions. D-branes require higher dimensionality of space-time.

We wish to look into the problem of black hole entropy in the special case of spherical symmetry. There are many advantages to this case. Firstly, spherical symmetry simplifies many technical problems but still allows enough degrees of freedom to be interesting and dynamical. Stationary black holes are fairly well understood. However, generally black holes will go through some dynamical period in which the horizon area increases or decreases as matter is accumulated into the black hole or as energy is radiated away, for instance by the Hawking process. It is this dynamical setting that we are most interested in studying. Secondly, in spherical symmetry, there is preferred time direction given by the Kodama vector [11]. In the stationary case,

the metric admits a time-like Killing vector from which we define time (an important step in quantizing a system). The Kodama vector is a natural analogue of a Killing vector [3] which can be used in a dynamical setting. Related to this choice of time, we can define a local energy where normally only a global energy can be defined. Thirdly, there is a dynamical local definition of the outer surface of a black hole. In stationary systems, the horizon is a Killing horizon, which coincides with both the event horizon and the apparent horizon. In a general case, the locally defined apparent horizon is not the same as the event horizon. So there arises a problem of which horizon gives the black hole entropy. In the case of spherical symmetry it is particularly simple to define a local horizon as a trapping horizon [12], a hypersurface foliated by marginal surfaces. This definition does not depend on any global characteristics such as the asymptotic behavior of the space-time and is a natural candidate for the outer surface of the back hole.

In this paper, we will look at two methods of defining black hole entropy for the case of spherical symmetry. In the second section, a reduced dimensional action is derived by integrating out the angular degrees of freedom on the sphere of symmetry, keeping the surface terms. It is from these surface terms that entropy is proposed to arise [13] [14]. In the dual-null form of the action, we indeed get an edge term that is proportional to the area, suggesting a possible dynamical definition of the entropy. In the third section, we look at the Noether currents of the action. In a n-form structure, Wald [4] found that the conserved charge associated with the Killing fields on the black hole when suitably rescaled gives the entropy of the stationary black hole. This method is easily generalizable to other actions. So along these lines, we begin by looking at the normal conserved charges of the spherically symmetric Einstein action. From these charges it is possible to identify the energy and write down an energy balance or first law from which an entropy can be defined [3]. Then, we generalize Wald's Noether charge to the trapping horizon for both the reduced action and the full four-dimensional action. The resulting dynamical conserved charge takes a similar form to the stationary case and reduces exactly to Wald's charge in the stationary limit. This charge when rescaled in the same way as the Wald charge gives an entropy that is one-quarter of the area of the trapping horizon in the case of Einstein gravity. In the derivation, it is not required to have a bifurcation surface in any sense. Then, in this dynamical setting, the question of what this conserved charge might be is addressed.

2 Reduced Action with Boundary Terms

In deriving the equation of motion from an action, a boundary term must sometimes be included to cancel a total derivative when integrating by parts. It has been proposed [13] that this surface term gives rise to the entropy of the black hole. It indeed gives the area law in the case of the Euclidean black hole [15]. It is these surface

terms which we wish to study in the spherically symmetric case where the boundary of the black hole can be defined dynamically as the trapping horizon. The resulting edge term in the reduced action defined below is proportional to the area. As such, it is a possible candidate for a dynamical definition of the entropy of a black hole.

In the case of spherical symmetry, it possible to write an effective two-dimensional action by integrating over the angular directions of the sphere. Let us look at this action and the resulting boundary term. For this derivation, we start with the action with boundary terms in four-dimensional space-time,

$$S = \int_M \sqrt{g} R^{(4)} d^4x + 2\int_{\partial M} \sqrt{g} K^{(4)} d^3x. \tag{1}$$

In the case of sphere symmetry, we can write the line element as

$$ds^2 = r^2 d^2\Omega + g_{ij} dx^i dx^j \tag{2}$$

where g^{ij} is the induced metric on the remaining two dimensional sub-manifold. The manifold is then foliated by spheres labelled by the coordinates $\{x_i\}$ which have an area A. The areal radius is then defined by $r = \sqrt{A/4\pi}$. The scalar curvature in four dimensions can be written in terms of the curvature of the two-dimensional normal sub-manifold and a term involving r,

$$R^{(4)} = R^{(2)} + \frac{(2 - 2g^{ij}\nabla_i\nabla_j(r^2) + 2g^{ij}\nabla_i r\nabla_j r)}{r^2}. \tag{3}$$

The boundary surface will also be chosen to respect the spherical symmetry, giving rise to a natural definition of the boundary on the sub-manifold. The normal to the surface in coordinate representation takes the form $n^a = \{0, 0, n^i(x)\}$, where the first two coordinates are the angular directions, and $n^i(x)$ is independent of the angular coordinates. With this definition, the extrinsic curvature can also be written in terms of its two-dimensional counterpart and the areal radius,

$$K^{(4)} = K^{(2)} + \frac{n^j\nabla_j(r^2)}{r^2}. \tag{4}$$

The terms in the above action (1) can be replaced by their two-dimensional counterparts suggesting that an effective or reduced dimensional action can be written on the two-dimensional normal manifold. We derive this action by integrating out the angular dependence, in essence averaging over them. Substituting in the above express for the curvature and integrating the angular directions, we get a reduced action in two dimensions,

$$\begin{aligned} S &= 2\pi \int_M \sqrt{g} \left(r^2 R^{(2)} + 2 - 2g^{ij}\nabla_i\nabla_j(r^2) + 2g^{ij}\nabla_i r\nabla_j r \right) d^2x \\ &\quad + 4\pi \int_{\partial M} \sqrt{g}\ r^2 K^{(2)} + n^j\nabla_j(r^2) d^1x. \end{aligned} \tag{5}$$

Note: the third term in the reduced action is just a total derivative and does not affect the dynamics. So removing this term as a boundary term, it cancels the second term in the surface action, leaving us with

$$S = 2\pi \int_M \sqrt{g} \left(r^2 R^{(2)} + 2 + 2g^{ij} \nabla_i r \nabla_j r \right) d^2x + 4\pi \int_{\partial M} \sqrt{g}\ r^2 K^{(2)} d^1x. \qquad (6)$$

This action takes a similar form as the string motivated dilaton action [16] where the areal radius replaces the dilaton ($r^2 = e^{-\phi}$). However, it contains a two-dimensional cosmological constant like term (the second term in (6)). This term breaks the conformal symmetry, which is to be expected since the original four-dimensional action was not conformally invariant.

For the reduced action (6), we get two sets of equations of motion, one from varying the two-dimensional metric and the other from varying r. As we will see, these equations of motion replicate the equations of motion coming for the original four-dimensional action (1). In the derivation[1] of the four-dimensional surface term $\int K$, the metric on the boundary is taken to be fixed. This means that the induced metric and r, which replaces $g_{\theta\theta}$ and $g_{\phi\phi}$ should also be fixed on the boundary, thus to be consistent we should also have Dirichlet boundary conditions for r. The reduced action with respect to the variation in the metric gives,

$$\begin{aligned} \delta S &= 2\pi \int_M \sqrt{g} \delta g^{ij} \left[-\frac{1}{2} g_{ij} (2 - 2g^{kl} \nabla_k r \nabla_l r + 4 g^{kl} r \nabla_k \nabla_l r) - 2r \nabla_i \nabla_j r \right] \\ &\quad + 2\pi \int_{\partial M} \sqrt{h} \left[(g^{ik} g^{jl} - g^{ij} g^{kl})(n_i r^2 \nabla_j (\delta g_{kl}) - n_j \nabla_i (r^2) \delta g_{kl}) \right] \\ &\quad + 4\pi \int_{\partial M} \sqrt{h} r^2 n^i h^{jk} \nabla_i (\delta g_{jk}), \end{aligned} \qquad (7)$$

where we have made use of the fact that $R_{ab} = 1/2 g_{ab} R$ in two dimensions. The metric h_{ab} is the induced metric on the one dimensional surface. For the boundary conditions $\delta g_{ab} = 0$ on the surface, the surface terms cancel. For the case that only $\delta h_{ab} = 0$, there exists a gauge transformation on the surface that will make $\delta g_{ab} = 0$, thus fixing part of the gauge on the surface. The volume term then gives the equations of motion,

$$\frac{1}{2} g_{ij} (2 - 2g^{kl} \nabla_k r \nabla_l r + 4 r g^{kl} \nabla_k \nabla_l r) - 2r \nabla_i \nabla_j r = 8\pi T^{(2)}_{ij} \qquad (8)$$

where $T^{(2)}_{ij}$ is the reduced stress-energy tensor of the matter fields.

Now let us consider the variation of the reduced action (6) with respect to the areal radius r.

$$\delta S = 2\pi \int_M \sqrt{g}\ \delta r \left[2R - 4 g^{ij} \nabla_i \nabla_j r \right] + 4\pi \int_{\partial M} \sqrt{h}\ \delta r \left[2K + 2 n^i \nabla_i r \right]. \qquad (9)$$

[1] See Wald [17] appendix E

For the boundary condition $\delta r = 0$, the surface term drops and we are left with the equation of motion from the volume term,

$$2rR - 4g^{ij}\nabla_i\nabla_j r = -4\pi\rho \tag{10}$$

where $\rho = \frac{1}{4\pi}\delta S_{\text{matter}}/\delta r$ which is related to the angular stress-energy terms from the original four-dimensional theory, as we will see below. To try to generalize to other boundary conditions is difficult because of the non-linear way in which the components of the metric tensor interact. To complete the equations of motion, we need to consider the form of the matter action in the reduced two-dimensional theory.

Of course it is possible to consider complicated and more general matter actions, however, to simplify matters, let us only consider diffeomorphism invariant actions that have the form $\mathcal{L}(\phi, \nabla\phi, g)$ where the ϕ is the set of matter fields. In this paper, we will not consider matter coupled to the curvature. Let us start with the Klein-Gordan action,

$$S_m = \int \sqrt{g}(g^{ab}\partial_a\phi\partial_b\phi + m^2\phi^2)d^4x. \tag{11}$$

For notational purposes, tensor and operators labelled by the beginning of the alphabet $(a, b, c, \ldots)$ will be four-dimensional, while their counterparts in two dimensions will be labelled by the middle of the alphabet $(i, j, k, \ldots)$. If the fields are not dependent on the angular coordinates, then the action is just,

$$S_m = 2\pi \int \sqrt{g^{(2)}}r^2(g^{ij}\partial_i\phi\partial_j\phi + m^2\phi^2)d^2x. \tag{12}$$

Generically, the action will take the form

$$S_m = 2\pi \int \sqrt{g}r^2\mathcal{L}' \tag{13}$$

With spherical symmetry, the four-dimensional stress-energy tensor takes block diagonal form, $T_{\theta\theta}d\theta^2 + T_{\phi\phi}d\phi^2 + T^{(4)}_{ij}dx^i dx^j$. The stress-energy tensor of the reduced action is related to the restriction of the four-dimensional stress energy by a scaling factor which is just the area,

$$T^{(2)}_{ij} = r^2\left[\partial_i\phi\partial_j\phi - \frac{1}{2}g_{ij}(g^{kl}\partial_k\phi\partial_l\phi + m^2\phi^2)\right] = 4\pi r^2 T^{(4)}_{ij} \tag{14}$$

To recover all the four-dimensional equations of motion, we must also consider variations with respect to r. This variation of the matter actions given us the definition of ρ (10),

$$\rho = 2r(g^{ij}\partial_i\phi\partial_j\phi + m^2\phi^2) = -4rT^\theta_\theta \tag{15}$$

In the above equations (8) and (10), we can see how the original stress-energy tensor enters into the reduced two-dimensional equation of motion. In this form it is easy to see that for a generic action, it will also take the same form. The matter action

is encoded into the two-dimensional stress-energy and ρ, which is the angular part of the stress-energy tensor.

The equations of motion are equivalent under diffeomorphisms, so we are free to choose any set of coordinates. The above equations of motion (8) and (10), greatly simplify in the case of double-null coordinates given by

$$g_{ij} = \begin{pmatrix} 0 & -e^{-f} \\ -e^{-f} & 0 \end{pmatrix}. \tag{16}$$

In this gauge choice, we get the same equations of motion as [18],

$$\partial_\pm\partial_\pm r + \partial_\pm f \partial_\pm r = -4\pi r T^{(4)}_{\pm\pm} \tag{17}$$

$$r\partial_+\partial_- r + \partial_+ r\partial_- r + \frac{1}{2}e^{-f} = 8\pi r^2 T^{(4)}_{+-} \tag{18}$$

$$r\partial_+\partial_- f - 2\partial_+\partial_- r = 8\pi r e^{-f} T^\theta_\theta \tag{19}$$

The areal radius r is related to the causal structure of the space-time. Following [12], a sphere is said to be untrapped, marginal or trapped as $\nabla^a r$ is spatial, null, or temporal. For $\nabla^a r$ future directed, the trapped (or marginal) surface is said to be future trapped, and likewise for past trapped surfaces. A hypersurface that is foliated by marginal surfaces is called a trapping horizon. A future outer trapping horizon is proposed to be the outer boundary of a black hole [18]. This definition is purely locally defined, and unlike event horizons or apparent horizons, it does not depend on any global conditions such as asymptotic flatness.

It seems natural to set boundary conditions for a dynamical black hole on the trapping horizon, which may be thought of as the inner boundary of the external space-time. Normally one fixes the outer boundary as past and future infinity ($\mathcal{J}^\pm$), with the boundary conditions being set by some asymptotic behavior. In particular, it is possible to determine a total energy of the system, the Bondi energy or (at spacelike infinity) the ADM energy [19]. Such energies are defined globally. However in the spherically symmetric case, the Misner-Sharp [20] energy can be defined as

$$E = \frac{1}{2}r(1 - \nabla_a r\nabla^a r), \tag{20}$$

which is a purely local statement. The various properties of this energy have been investigated in various limiting cases [18]. It short, it represents a local active gravitational energy. On the trapping horizon, $\nabla^a r$ is null. Therefore, on the horizon, $2E = r$, which generalizes the normal definition of the radius of the Schwarzschild event horizon to dynamic black holes.

With these local definitions of the boundary of a black hole and gravitational energy, we would like to further investigate the entropy of this system. As stated before, it is believed that one of the possible sources of entropy comes from the

surface term in the above action (1) [15]. Following these lines, let us study the boundary in the reduced action, using a trapping horizon as the inner boundary of the space-time (the outer boundary of the black hole). This hypersurface is typically spacelike for a dynamic black hole, but it becomes null for a stationary black hole. For a null hypersurface the normal vector becomes tangent to the hypersurface, so that the usual 3+1 definition of extrinsic curvature becomes an intrinsic function of the hypersurface. For this reason and others, it is easier to process in a double-null formulation.

Generalized Lagrangian and Hamiltonian theories of double-null systems were developed in Ref. [21] and applied to the Einstein system in Ref. [22]. The basic idea is to have an action

$$S_{DN} = \int L\, dx^+ dx^- \tag{21}$$

where the Lagrangian L is a function of some configuration fields q and two corresponding Lie derivatives $L_\pm q$, which in the current case reduce to the partial derivatives $\partial_\pm q = \partial q/\partial x^\pm$. The Lagrangian for vacuum Einstein theory given in [22] evaluates to

$$L = \partial_+ A \partial_- f + \partial_- A \partial_+ f - A^{-1}\partial_+ A \partial_- A + 8\pi e^{-f} \tag{22}$$

in spherical symmetry. One may check that the corresponding Euler-Lagrange equations yield the G_{+-} and G^θ_θ components of the Einstein equation, with the $G_{\pm\pm}$ components requiring Lagrange multipliers. This Lagrangian was obtained from the Einstein-Hilbert Lagrangian

$$S_{EH} = \int_M \sqrt{g} R^{(4)} d^4x \tag{23}$$

by removing boundary terms, which may be recovered using the expression (3) for $R^{(4)}$ and noting $R^{(2)} = -2e^f \partial_+\partial_- f$, yielding

$$S_{EH} = \int \left(4\partial_+\partial_- A - 2A\partial_+\partial_- f - A^{-1}\partial_+ A \partial_- A + 8\pi e^{-f}\right) dx^+ dx^-. \tag{24}$$

A comparison allows one to write

$$S_{DN} = S_{EH} + S_+ + S_- + S_0 \tag{25}$$

where we have separated three boundary terms:

$$S_\pm = \int A \partial_\pm f dx^\pm \tag{26}$$

are obtained by integrating total derivatives in $\partial_\mp$, but there is also a double total derivative in $\partial_+\partial_-$ which may be integrated twice to

$$S_0 = -4A. \tag{27}$$

The fact that this double boundary term is basically the area suggests a connection with entropy. Comparing with the $3+1$ ADM formalism in [13], we also see their entropy is related to an edge term. The other boundary terms $S_\pm$ are gauge-dependent and can be set to zero by taking $x^\pm$ to be affine parameters, $\partial_\pm f = 0$, on the null hypersurfaces $x^\mp = 0$. Consequently a suggestion for defining entropy is

$$-\frac{S_0}{16}. \tag{28}$$

Note that this makes sense in the dynamical case, agreeing with the usual expression $A/4$, and may be generalized to other theories of gravity in which a similar decomposition of the dual-null action occurs.

Alternatively, since gauge dependence has been shown [9] to lead to observables on the edge, so called edge states, it may be interesting to study the gauge dependent surface terms $S_\pm$. However, it is unclear if the above double-null formulation is complete for two reasons. The first is that the gauge transformation of the null coordinates are only a small subset of the original gauge transformations. Related to this, the second reason is that the equations of motions above (17)-(19) should all be reproduced in the double null formulation. Namely, the equation (17) most be included as a Lagrange multiplier term. This term might also lead to an edge observables.

3 Noether Currents

Next, we would like to consider the Noether currents of the reduced action and the original four-dimensional action. In Wald and Iyer's work [23], they have defined an entropy in terms of the Noether current. We would like to compare Wald's currents with the known conversed currents of the spherically symmetric theory and the generalized entropy discussed in [3].

For every symmetry, there is an associated conserved current (Noether theorem). Gravitational theories are generally defined from a diffeomorphism invariant action. These diffeomorphisms are locally generated by an arbitrary vector field ξ^a. So for each of these vector fields there is an associated $(n-1)$ form Noether current and a $(n-2)$ form Noether charge, where n is the dimension of the manifold on which the action is defined. Wald defined an entropy for a stationary black in terms of the integral of the Noether charge associated with the horizon Killing fields on the bifurcation surface [4]. In this definition, it is required that the surface gravity be normalized to unity.

The Noether current is defined in terms of the symplectic potential Θ and the n-form action L [4],

$$J = \Theta(\phi, \mathcal{L}_\xi \phi) - \xi \cdot L \tag{29}$$

where ξ is the generator of the diffeomorphism. The symplectic potential form is defined from variation of the action,

$$\delta L = \mathcal{E}\delta\phi + d\Theta, \tag{30}$$

where the ϕ is the dynamic fields and $\mathcal{E}$ is the equations of motion. The potential Θ is only defined up to a total derivative.[2]

Quoting the results for the pure gravitational action [23], the Noether current and charge are

$$J_{abc} = \frac{1}{8\pi}\epsilon_{dabc}\nabla_e\left(\nabla^{[e}\xi^{d]}\right) \tag{31}$$

$$Q_{ab} = -\frac{1}{16\pi}\epsilon_{abcd}\nabla^c\xi^d \tag{32}$$

where the brackets indicate the anti-symmetric sum with the convention of Wald [17]. For the diffeomorphism generated by the Killing vector ξ^a which generates the Killing horizon,

$$\epsilon_{abcd}\nabla^c\xi^d = -2\kappa\eta_{ab} \tag{33}$$

where κ is the surface gravity, η_{ab} is the area form of the spatial surfaces lying in the horizon, and the space-time volume form is $\epsilon_{abcd} = 2\epsilon_{[ab}\eta_{cd]}$. Thus the integral of the Noether charge over such a bifurcation surface is

$$\oint Q = \frac{\kappa A}{8\pi}. \tag{34}$$

Then the entropy is defined as 2π times this charge with the surface gravity being normalized by $\kappa = 1$, giving the standard stationary black hole entropy of quarter of the area.

Before looking at this in the spherically symmetric case, let us look at the known conserved currents of this system. There are two conserved currents for the spherical system. The first is the Kodama vector. It is defined as a curl of the areal radius, $k^a = \epsilon^{ab}\nabla_b r$. The time-like Killing vector that generates time translations can be replaced by the Kodama vector in the spherically symmetric case [3]. It follows from the above definition that

$$k^a\nabla_a r = 0 \tag{35}$$

$$k^a k_a = \frac{2E}{r} - 1 \tag{36}$$

giving the relation between the local energy and this time like vector. It also follows that k is a conserved current,

$$\nabla_a k^a = 0, \tag{37}$$

[2] See [23] for a discussion of methods to choose a given form of the symplectic potential.

with conserved charge given by the Gauss theorem,

$$V = -\int_{\Sigma} k^a d\Sigma_a \tag{38}$$

where $d\Sigma^a$ is the volume form times a future directed unit normal vector of the space-like hypersurface Σ. This charge is the areal volume $V = \frac{4}{3}\pi r^3$.

From the stress-energy tensor we can define two invariants; the work density,

$$w = -\frac{1}{2}T^{ij}g_{ij} \tag{39}$$

and the energy flux (localized Bondi flux)

$$\psi^a = T^{ab}\nabla_b r + w\nabla^a r. \tag{40}$$

The energy flux may be replaced by a energy-momentum density along the Kodama vector, $j^a = T^{ab}k_b$ or equivalently

$$j^a = \epsilon^{ab}\psi_b + wk^a \tag{41}$$

This is also a conserved current,

$$\nabla_a j^a = 0 \tag{42}$$

which has a conserved charge equal to the energy,

$$E = -\int_{\Sigma} j^a d\Sigma_a. \tag{43}$$

A surface gravity for dynamic black holes can be defined [3] from the Kodama vector by noting an analogous definition with the Killing vector ,

$$\xi^b\nabla_{[b}\xi_{a]} = \kappa\xi_a. \tag{44}$$

Replacing the Killing vector with the Kodama vector, we get from the equations of motion

$$k^b\nabla_{[b}k_{a]} = (E/r^2 - 4\pi r w)k_a = \kappa k_a \tag{45}$$

on a trapping horizon. Alternatively, we may define the surface gravity directly from the Kodama vector as

$$\kappa = \epsilon^{ab}\nabla_a k_b/2. \tag{46}$$

Summing up, we have two kinematical quantities (r, k^a), two gravitational quantities (E, κ) and two matter quantities (w, ψ^a) or (w, j^a). For these definitions, it is possible to define an entropy strictly in a thermodynamical setting [25] in the same way that it was first introduced by Clausius [26]. From the equations of motion, we can write an energy balance equation or first law

$$\nabla_a E = A\psi_a + w\nabla_a V \tag{47}$$

The second term on the right side can be thought of as a type of work, and the first term as a energy flux term, analogous to the heat flux term in standard thermodynamics. Using the above definition of the surface gravity and the equations of motion, we can rewrite this energy flux as

$$A\psi_a = \frac{\kappa \nabla_a A}{8\pi} + r\nabla_a \left(\frac{E}{r}\right) \tag{48}$$

On any surface where E/r is constant, this last term disappears. This is the case on a trapping horizon, where $\nabla^a r$ is null and $2E = r$. Identifying the temperature with $\kappa/2\pi$, the entropy associated with the black hole trapping horizon is then $A/4$.

From the conserved currents and the equations of motion, we are able to identify a first law of thermodynamics which enables us to derive the expected area law for the entropy. This definition works in a dynamical setting. In order to compare results, we wish to generalize Wald's results (34) to this dynamical setting where the time-like Killing vector is replaced by the Kodama vector and the surface is defined as the trapping horizon.

So returning to Wald's define of entropy, let us look at the Noether currents of spherically symmetric gravity. We have two actions to consider, the reduced action and the original four dimensional action. At this point, we switch to Wald's scaling of the Lagrangian $l = \epsilon R^{(4)}/16\pi$. Let us start with the reduced action (6). From the derivation of the equations of motion (7), the symplectic potential can be seen to take the form,

$$\begin{aligned}\Theta_i &= \frac{1}{8}\epsilon_{ij}\left[(g^{jl}g^{km} - g^{jk}g^{lm})(r^2\nabla_l(\delta g_{km}) - \nabla_j(r^2)\delta g_{km}) + 4\nabla_i r\delta r\right] \\ &\quad + \text{matter terms.}\end{aligned} \tag{49}$$

The variation be generated by the diffeomorphism given are by $\delta g_{ij} = \nabla_i\xi_j + \nabla_j\xi_i$ and $\delta r = \nabla_i r\xi^i$. For which the current can be written,

$$\begin{aligned}j_i &= \frac{1}{4}\epsilon_{ij}\Big[\nabla_k\left(r^2\nabla^{[j}\xi^{k]} - 2\xi^{[j}\nabla^{k]}(r^2)\right) \\ &\quad -\xi_k\left(g^{jk}(2 - 2\nabla_l r\nabla^l r + 4r\nabla_l\nabla^l r) - 4r\nabla^j\nabla^k r\right)\Big] + \text{ matter terms}\end{aligned} \tag{50}$$

For the case of pure gravity or the matter fields governed by a Klein-Gordon action, the matter fields can be replaced by the gravitational fields by use of the equations of motion (8). This exchange cancels many of the terms, resulting in a conserved current that is equivalent to the dilaton action derived by Wald in [23] with e^ϕ being replaced by r^2,

$$j_i = \frac{1}{4}\epsilon_{ij}\nabla_k(r^2\nabla^{[j}\xi^{k]} - 2\xi^{[j}\nabla^{k]}(r^2)), \tag{51}$$

The corresponding conserved charge is then

$$q = -\frac{1}{8}\epsilon_{ij}(r^2\nabla^i\xi^j - 2\xi^i\nabla^j(r^2)). \tag{52}$$

If we started from the original four-dimensional action, we get a slightly different form for this conserved current and charge,

$$J_{iab} = \frac{1}{8\pi}\eta_{ab}\epsilon_{ij}(r^2\nabla_k\nabla^{[j}\xi^{k]} - 2r\nabla_k\xi^{[j}\nabla^{k]}r), \tag{53}$$

$$Q_{ab} = -\frac{1}{16\pi}\eta_{ab}\epsilon_{ij}r^2\nabla^i\xi^j \tag{54}$$

where η_{ab} is now the area form of the unit sphere and ξ is restricted to the tangent space of the reduced two-dimensional space. We can compare these terms by noting that

$$J_{iab} \cong \frac{1}{2\pi}\eta_{ab}j_i \qquad\qquad Q_{ab} \cong \frac{1}{2\pi}\eta_{ab}q \tag{55}$$

should be equivalent up to total derivatives. The difference is due to the fact that we have removed a surface term $\nabla^a\nabla_a r^2$ in the reduced action. However, the current and charge are only defined up to a total derivative anyway [4]. This once again brings up an issue that Iyer and Wald have tried to address [23]. With this additional term, the entropy can be changed resulting in a different entropy. We will assume that the correct action is the one with the surface term still included and continue with the evaluation of the original four-dimensional action.

As already emphasized, the Kodama vector is a natural candidate to replace the Killing vector as the generator of the relevant diffeomorphism on the horizon of a dynamic black hole. Using the Kodama vector to generate the diffeomorphism, the current is

$$J_{iab} = \frac{1}{8\pi}\eta_{ab}\epsilon_{ij}(r^2\nabla_k\nabla^{[j}\xi^{k]} - 2r\nabla_k\xi^{[j}\nabla^{k]}r) \tag{56}$$

$$= -\frac{1}{8\pi}\eta_{ab}\nabla_i(r^2\nabla_j\nabla^j r) \tag{57}$$

The conserved charge from the Kodama vector is then given by

$$Q = \frac{1}{16\pi}\eta_{ab}\left(r^2\nabla_j\nabla^j r\right) \tag{58}$$

Integrating this on the trapping horizon,

$$\oint Q = \frac{\kappa A}{8\pi} \tag{59}$$

where κ is the dynamical surface gravity (46). In the static case, the surface gravity is a constant on the horizon and can be rescaled to be unity and Wald's entropy is recovered.

The question that comes to mind is "What is this conserved charge?" The normal conserved current associated with time translations is an energy current. By replacing κ with (46), we see that

$$\oint 2Q = E - 3wV \tag{60}$$

which seems to be an energy rather than entropy in general. In the vacuum case, w is zero and drops leaving just the energy.

Note also that rescaling the surface gravity to unity is effectively assuming that $\delta\kappa = 0$, implying that

$$\delta \oint Q = \frac{\kappa \delta A}{8\pi}. \tag{61}$$

The first law of the Schwarzschild black hole also takes this same form $\delta m = \kappa\delta A/8\pi$. This again suggests that this conserved charge has more to do with an energy than an entropy. However, in [27], a solution to this problem has been suggested.

With the Wald-Iyer Noether current definition of the entropy now adapted to a dynamical setting without the requirement of a bifurcation surface, we can ask questions of how the matter fields may influence this definition, whereas before only the stationary case could be studied because of the need for a Killing field. However, in the above calculations, only the vacuum and scalar field case were studied. In the vacuum case, the current has an extra term that is not written down in Wald's paper, which is just the Einstein curvature. On shell, this is of course just zero and doesn't affect anything. In the scalar field case, the term from the matter fields is the stress-energy tensor. So with the Einstein equations this terms again cancels. The question is what happens with other forms of matter? Some initial calculations suggest that other terms will appear in the current.

In the last two sections, we have investigated two possible sources of entropy. With the introductions of the trapping horizon that is easy to define in the spherically symmetric case, we see that in both cases of boundary terms and Noether currents there is a possible dynamical definition of the entropy of a black hole. This entropy is just one-quarter of the area of the trapping horizon. This satisfies a dynamical area law in agreement with the generalized first law that has also been found in the spherical case [3]. However, these results need to be compared with a statistical-mechanical definition of entropy coming from a quantum theory, which as yet has only been done in the stationary case e.g. [28] or in a reduced dimensional case e.g [10]. Perhaps the reduced dimensional action (6) above may be simple enough to quantize and how this affects the definition of the entropy is unclear.

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